

A Graph Reduction Step Preserving Element-Connectivity and Applications

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Abstract

Given an undirected graph $G = (V, E)$ and subset of terminals $T \subseteq V$, the *element-connectivity* $\kappa'_G(u, v)$ of two terminals $u, v \in T$ is the maximum number of u - v paths that are pairwise disjoint in both edges and non-terminals $V \setminus T$ (the paths need not be disjoint in terminals). Element-connectivity is more general than edge-connectivity and less general than vertex-connectivity. Hind and Oellermann [21] gave a graph reduction step that preserves the *global* element-connectivity of the graph. We show that this step also preserves *local* connectivity, that is, all the pairwise element-connectivities of the terminals. We give two applications of this reduction step to connectivity and network design problems.

- Given a graph G and disjoint terminal sets T_1, T_2, \dots, T_m , we seek a maximum number of element-disjoint Steiner forests where each forest connects each T_i . We prove that if each T_i is k element connected then there exist $\Omega(\frac{k}{\log h \log m})$ element-disjoint Steiner forests, where $h = |\bigcup_i T_i|$. If G is planar (or more generally, has fixed genus), we show that there exist $\Omega(k)$ Steiner forests. Our proofs are constructive, giving poly-time algorithms to find these forests; these are the first non-trivial algorithms for packing element-disjoint Steiner Forests.
- We give a very short and intuitive proof of a spider-decomposition theorem of Chuzhoy and Khanna [12] in the context of the single-sink k -vertex-connectivity problem; this yields a simple and alternative analysis of an $O(k \log n)$ approximation.

Our results highlight the effectiveness of the element-connectivity reduction step; we believe it will find more applications in the future.

1 Introduction

In this paper we consider several connectivity and network design problems. Given an undirected graph G and two nodes u, v we let $\lambda_G(u, v)$ and $\kappa_G(u, v)$ denote the edge and vertex connectivities between u and v in G . It is well-known that edge-connectivity problems are “easier” than their vertex-connectivity counterparts. Vertex-connectivity exhibits less structure than edge-connectivity and this often translates into significant differences in the algorithmic and computational difficulty of the corresponding problems. As an example, consider the well-known survivable network design problem (SNDP): the input consists of an undirected edge-weighted graph G and connectivity requirements $r : V \times V \rightarrow Z^+$ between each pair of vertices. The goal is to find a min-cost subgraph H of G such that each pair u, v has $r(u, v)$ disjoint paths between them in H . If the paths are required to be edge-disjoint ($\lambda_H(u, v) \geq r(u, v)$) then the problem is referred to as EC-SNDP and if the paths

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are required to be vertex-disjoint the problem is referred to as VC-SNDP. Jain [23] gave a 2-approximation for EC-SNDP based on the powerful iterated rounding technique. On the other hand, VC-SNDP is known to be hard to within polynomial factors [28, 4]. To address this gap, Jain *et al.* [25] introduced a connectivity measure intermediate to edge and vertex connectivities known as *element-connectivity*. The vertices are partitioned into terminals $T \subseteq V$ and non-terminals $V \setminus T$. The element-connectivity between two terminals u, v , denoted by $\kappa'_G(u, v)$ is defined to be the maximum number of paths between u and v that are pairwise disjoint in edges and non-terminals (the paths can share terminals). In some respects, element-connectivity resembles edge-connectivity: For example, $\kappa'(u, w) \geq \min(\kappa'(u, v), \kappa'(v, w))$ for any three terminals u, v, w ; this triangle inequality holds for edge-connectivity but does not for vertex-connectivity. In element-connectivity SNDP (ELC-SNDP) the requirements are only between terminals and the goal is to find a min-cost subgraph H such that $\kappa'_H(u, v) \geq r(u, v)$ for each $u, v \in T$. Fleischer, Jain and Williamson [16] (see also [11]) generalized the iterated rounding technique of Jain for EC-SNDP to give a 2-approximation for ELC-SNDP. In other respects, element-connectivity is related to vertex connectivity. One class of problems motivating this paper is on generalizing the classical theorem of Menger on s - t vertex-connectivity; we discuss this below.

In studying element-connectivity, we often assume without loss of generality that there are no edges between terminals (by subdividing each such edge) and hence $\kappa'(u, v)$ is the maximum number of non-terminal disjoint u - v paths. Menger's theorem shows that the maximum number of internally vertex-disjoint s - t paths is equal to $\kappa(s, t)$. Hind and Oellermann [21] considered a natural generalization to multiple terminals. Given a terminal set $T \subseteq V$, what is the maximum number of trees that each contain T and are disjoint in $V \setminus T$? The natural upper bound here is the element connectivity of T in G , in other words, $k = \min_{u, v \in T} \kappa'(u, v)$. In [21] a graph reduction step was introduced to answer this question. Cheriyan and Salavatpour [9] called this the problem of packing element-disjoint Steiner trees; crucially using the graph reduction step, they showed that there always exist $\Omega(k / \log |T|)$ element-disjoint Steiner trees and moreover, this bound is tight (up to constant factors) in the worst case. In contrast, if we seek edge-disjoint Steiner trees then Lau [32] has shown that if T is $26k$ edge-connected in G , there are k edge-disjoint trees each of which spans T .

Finally, we remark that in some recent work Chuzhoy and Khanna [12] gave an $O(k \log |T|)$ approximation for the special case of VC-SNDP in which a terminal set T needs to be k -vertex-connected (this is equivalent to the single-sink problem). Their algorithm and analysis are based on a structural characterization of feasible solutions — they use element-connectivity (they call it weak connectivity) as a key stepping stone. Subsequent to this paper, Chuzhoy and Khanna [13] gave a simple and elegant reduction from the the *general* VC-SNDP problem to ELC-SNDP, obtaining an $O(k^3 \log n)$ -approximation and reinforcing the connection between element- and vertex-connectivity.

The discussion above suggests that it is fruitful to study element-connectivity as a way to generalize edge-connectivity and attack problems on vertex-connectivity. In this paper we consider the graph reduction step for element-connectivity introduced by Hind and Oellermann [21] (and rediscovered by Cheriyan and Salavatpour [9]). We generalize the applicability of the step and demonstrate applications to several problems.

A Graph Reduction Step Preserving Element Connectivity: The well-known *splitting-off* operation introduced by Lovász [34] is a standard tool in the study of (primarily) edge-connectivity problems. Given an undirected multi-graph G and two edges su and sv incident to s , the splitting-off operation replaces su and sv by the single edge uv . Lovász proved the following theorem on splitting-off to preserve *global* edge-connectivity.

Theorem 1.1 (Lovász). *Let $G = (V \cup \{s\}, E)$ be an undirected multi-graph in which V is k -edge-connected for some $k \geq 2$ and degree of s is even. Then for every edge su there is another edge sv such that V is k -edge-connected after splitting-off su and sv .*

Mader strengthened the above theorem to show the existence of a pair of edges incident to s that when split-off preserve the *local* edge-connectivity of the graph.

Theorem 1.2 (Mader [35]). *Let $G = (V \cup \{s\}, E)$ be an undirected multi-graph, where $\deg(s) \neq 3$ and s is not incident to a cut edge of G . Then s has two neighbours u and v such that the graph G' obtained from G by replacing su and sv by uv satisfies $\lambda_{G'}(x, y) = \lambda_G(x, y)$ for all $x, y \in V \setminus \{s\}$.*

Generalization to directed graphs are also known [35, 17, 26]. The splitting-off theorems have numerous applications in graph theory and combinatorial optimization. See [34, 18, 31, 24, 6, 32, 33, 27] for various pointers and applications. Although splitting-off techniques can be sometimes be used in the study of vertex-connectivity, their use is limited and no generally applicable theorem akin to Theorem 1.2 is known. On the other hand, Hind and Oellermann [21] proved an elegant theorem on preserving global element connectivity. In the sequel we use $\kappa'_G(S)$ to denote $\min_{u, v \in S} \kappa'_G(u, v)$ and G/pq to denote the graph obtained from G by contracting vertices p, q .

Theorem 1.3 (Hind & Oellermann [21]). *Let $G = (V, E)$ be an undirected graph and $T \subseteq V$ be a terminal-set such that $\kappa'_G(T) \geq k$ for each pair $u, v \in T$. Let (p, q) be any edge where $p, q \in V \setminus T$. Then $\kappa'_{G_1}(T) \geq k$ or $\kappa'_{G_2}(T) \geq k$ where $G_1 = G - pq$ and $G_2 = G/pq$.*

This theorem has been used in two applications on element-connectivity [9, 27]. We generalize it to handle local connectivity, increasing its applicability.

Reduction Lemma. *Let $G = (V, E)$ be an undirected graph and $T \subseteq V$ be a terminal-set. Let (p, q) be any edge where $p, q \in V \setminus T$ and let $G_1 = G - pq$ and $G_2 = G/pq$. Then one of the following holds: (i) $\forall u, v \in T$, $\kappa'_{G_1}(u, v) = \kappa'_G(u, v)$ (ii) $\forall u, v \in T$, $\kappa'_{G_2}(u, v) = \kappa'_G(u, v)$.*

Remark 1.4. *The Reduction Lemma, applied repeatedly, transforms a graph into another graph in which the non-terminals form a stable set. Moreover, the reduced graph is a minor of the original graph.*

We give applications of the Reduction Lemma (using additional ideas) to two problems that we had briefly alluded to already. We discuss these below.

Packing Element-Disjoint Steiner Trees and Forests: There has been much interest in the recent past on algorithms for (integer) packing of disjoint Steiner trees in both the edge and element-connectivity settings [31, 24, 32, 33, 8, 9, 6]. (A *Steiner tree* is simply a tree containing the entire terminal set T .) See [20] for applications of Steiner tree packing to VLSI design. An outstanding open problem is Kriesell's conjecture which states that if the terminal set T is $2k$ -edge-connected then there are k -edge-disjoint Steiner trees each of which spans T ; this would generalize a classical theorem of Nash-Williams and Tutte on edge-disjoint spanning trees. Lau made substantial progress [32] and proved that $26k$ -connectivity suffices for k edge-disjoint Steiner trees; he extended his result for packing Steiner forests [33]. We remark that Mader's splitting-off theorem plays an important role in Lau's work. The element-disjoint Steiner tree packing problem was first considered by Hind and Oellermann. As we mentioned, Cheriyan and Salavatipour [9] gave a nearly tight bound for this problem. Their result relies crucially on Theorem 1.3 followed by a simple randomized coloring algorithm whose analysis extends a similar algorithm for computing the domatic number of a graph [15]. In [3] the random coloring idea was shown to apply more generally in the context of packing bases of an arbitrary monotone submodular function; in addition, a derandomization was provided in [3] via the use of min-wise independent permutations. It is also known that the problem of packing element-disjoint Steiner trees is hard to approximate to within an $\Omega(\log n)$ factor [8]. Here, we consider the more general problem of packing Steiner forests that was posed by [9]. The input consists of a graph $G = (V, E)$ and disjoint terminal sets T_1, T_2, \dots, T_m , such that $\kappa'_G(T_i) \geq k$ for $1 \leq i \leq m$. What is the maximum number of element disjoint forests such that in each forest T_i is connected for $1 \leq i \leq m$? Our local connectivity reduction step is primarily motivated by this question. For general graphs we prove that there exist $\Omega(k/(\log |T| \log m))$ element disjoint forests, where $T = \bigcup_i T_i$. This can also be viewed as an $O(\log |T| \log m)$ approximation for the problem. We apply the Reduction Lemma to obtain a graph in which the non-terminals are a stable set. We cannot however apply the random coloring

approach directly — in fact we can show that it does not work. Instead we decompose the graph into highly connected subgraphs and then apply the random coloring approach in each subgraph separately.

We also study the packing problem in planar graphs and graphs of fixed genus, and prove substantially stronger results. Here too, the first step is to use the Reduction Lemma (recall that the reduced graph is a minor of the original graph and hence is also planar). After the reduction step, we employ a very different approach from the one for general graphs. Our main insight is that planarity restricts the ability of non-terminals to provide high element-connectivity to the terminals. We formalize this intuition by showing that there are some two terminals u, v that have $\Omega(k)$ parallel edges between them which allows us to contract them and recurse. Using these ideas, for planar graphs we prove that there exist $\lceil k/5 \rceil - 1$ disjoint forests. Our method also extends to give an $\Omega(k)$ bound for graphs of a fixed genus, and we conjecture that one can find $\Omega(k)$ disjoint forests in graphs excluding a fixed minor; we give evidence for this by proving it for packing Steiner trees in graphs of fixed treewidth. Note that these bounds also imply corresponding approximation algorithms for maximizing the number of disjoint forests. These are the first non-trivial bounds for packing element-disjoint Steiner forests in general graphs or planar graphs. Since element-connectivity generalizes edge-connectivity, our bounds in planar graphs are considerably stronger than those of given by Lau [32, 33] for *edge*-connectivity. Our proof is simple, however, we remark that the simplicity of the proof comes from thinking about element-connectivity (using the Reduction Lemma) instead of edge-connectivity! Our proof also gives the strong property that the non-terminals in the forests all have degree 2.

Single-Sink k -vertex-connectivity: Polynomial factor inapproximability results for VC-SNDP [28, 4] have focused attention on restricted, yet useful, special cases of the problem. In recent work Chakraborty, Chuzhoy and Khanna [4] considered the single-sink k -vertex-connectivity problem for small k ; the goal is to k -vertex-connect a set of terminals T to a given root r . This problem is approximation-equivalent to the subset k -connectivity problem in which T needs to be k -connected [4]. If $k = 1$, this is the NP-Hard Steiner tree problem and a 2-approximation is well-known. For $k = 2$, a 2-approximation follows from [16] whose algorithm can handle the more general VC-SNDP with requirements in $\{0, 1, 2\}$. For $k > 2$ the first non-trivial approximation algorithm was given in [4]; the approximation ratio was $k^{O(k^2)} \log^4 n$. Improvements were given in [12, 5] with Chuzhoy and Khanna [12] achieving the currently best known approximation ratio of $O(k \log |T|)$. The algorithms are essentially the same in [4, 12, 5] and build upon the insights from [4]; the analysis in [12] relied on a beautiful decomposition result for k -connectivity which is independently interesting from a graph theoretic view point. The proof of this theorem in [12] is long and complicated although it is based on only elementary operations. Using the Reduction Lemma, we give an alternate proof of the main technical result which is only half a page long! We mention that the decomposition theorem has applications to more general network design problems such as the rent-or-buy and buy-at-bulk network design problems as shown in [5]. Due to space constraints we omit these applications in this paper.

Related Work: We have already mentioned most of the closely related papers. Our work on packing Steiner forests in planar graphs was inspired by a question by Joseph Cheriyan [7]. Independent of our work, Aazami, Cheriyan and Jampani [1] proved that if a terminal set T is k -element-connected in a planar graph then there exist $k/2 - 1$ element-disjoint Steiner trees, and moreover this is tight. They also prove that it is NP-hard to obtain a $(1/2 + \varepsilon)$ approximation for this problem. Our bound for packing Steiner Trees in planar graphs is slightly weaker than theirs; however, our algorithms and proofs are simple and intuitive, and generalize to packing Steiner forests. Their algorithm uses Theorem 1.3, followed by a reduction to a theorem of Frank *et al.* [19] that uses Edmonds’ matroid partition theorem. One could attempt to pack Steiner forests using their approach (with the stronger Reduction Lemma in place of Theorem 1.3), but the theorem of [19] does not have a natural generalization for Steiner forests. The techniques of both [1] and this paper extend to graphs of small genus or treewidth; we discuss this further in Section 3.2. We refer the reader to [4, 12, 5] for more discussion of recent work on single-sink vertex connectivity, including hardness results [4] and extensions to related problems such as the node-weighted case [12] and buy-at-bulk network design [5]. Nutov [36] has

recently given alternate algorithms, based on the primal-dual method, for the single-sink vertex-connectivity network design with approximation ratios comparable to those from [12]. These algorithms do not have the advantage of the structural decomposition of [12]. We mention that if $T = V$, that is, we wish to find a min-cost subgraph of G that is k -connected then an $O(\log^2 k)$ approximation is known [14, 30, 10]. We also refer the reader to a survey on network design by Kortsarz and Nutov [29].

2 The Reduction Lemma

Let $G(V, E)$ be a graph, with a given set $T \subseteq V(G)$ of terminals. For ease of notation, we subsequently refer to terminals as *black* vertices, and non-terminals (also called Steiner vertices) as *white*. The elements of G are white vertices and edges; two paths are *element-disjoint* if they have no white vertices or edges in common. Recall that the element-connectivity of two black vertices u and v , denoted by $\kappa'_G(u, v)$, is the maximum number of element-disjoint (that is, disjoint in edges and white vertices) paths between u and v in G . We omit the subscript G when it is clear from the context.

For this section, to simplify the proof, we will assume that G has no edges between black vertices; any such edge can be subdivided, with a white vertex inserted between the two black vertices. It is easy to see that two paths are element-disjoint in the original graph iff they are element-disjoint in the modified graph. Thus, we can say that paths are element disjoint if they share no white vertices, or that u and v are k -element-connected if the smallest set of white vertices whose deletion separates u from v has size k .

Recall that our lemma generalizes Theorem 1.3 on preserving global connectivity. We remark that our proof is based on a cutset argument unlike the path-based proofs in [21, 9] for the global case.

Reduction Lemma. *Given $G(V, E)$ and T , let $pq \in E(G)$ be any edge such that p and q are both white. Let $G_1 = G - pq$ and $G_2 = G/pq$ be the graphs formed from G by deleting and contracting pq respectively. Then, (i) $\forall u, v \in T, \kappa'_{G_1}(u, v) = \kappa'_G(u, v)$ or (ii) $\forall u, v \in T, \kappa'_{G_2}(u, v) = \kappa'_G(u, v)$.*

Proof: Consider an arbitrary edge pq . Deleting or contracting an edge can reduce the element-connectivity of a pair by at most 1. Suppose the lemma were not true; there must be pairs s, t and x, y of black vertices such that $\kappa'_{G_1}(s, t) = \kappa'_G(s, t) - 1$ and $\kappa'_{G_2}(x, y) = \kappa'_G(x, y) - 1$. The pairs have to be distinct since it cannot be the case that $\kappa'_{G_1}(u, v) = \kappa'_{G_2}(u, v) = \kappa'_G(u, v) - 1$ for any pair u, v . (To see this, if one of the $\kappa'_G(u, v)$ u - v paths uses pq , contracting the edge will not affect that path, and will leave the other paths untouched. Otherwise, no path uses pq , and so it can be deleted.). Note that one of s, t could be the same vertex as one of x, y ; for simplicity we will assume that $\{s, t\} \cap \{x, y\} = \emptyset$, but this does not change our proof in any detail. We show that our assumption on the existence of s, t and x, y with the above properties leads to a contradiction. Let $\kappa'_G(s, t) = k_1$ and $\kappa'_G(x, y) = k_2$. We use the following facts several times.

1. Any cutset of size less than k_1 that separates s and t in G_1 cannot include p or q . (If it did, it would also separate s and t in G .)
2. $\kappa'_{G_1}(x, y) = k_2$ since $\kappa'_{G_2}(x, y) = k_2 - 1$.

We define a vertex tri-partition of a graph G as follows: (A, B, C) is a vertex tri-partition of G if A, B , and C partition $V(G)$, B contains only white vertices, and there are no edges between A and C . (That is, removing the white vertices in B disconnects A and C .)

Since $\kappa'_{G_1}(s, t) = k_1 - 1$, there is a vertex-tri-partition (S, M, T) such that $|M| = k_1 - 1$ and $s \in S$ and $t \in T$. From Fact 1 above, M cannot contain p or q . For the same reason, it is also easy to see that p and q cannot be both in S (or both in T); otherwise M would be a cutset of size $k_1 - 1$ in G . Therefore, assume w.l.o.g. that $p \in S, q \in T$.

Similarly, since $\kappa'_{G_2}(x, y) = k_2 - 1$, there is a vertex-tri-partition (X, N', Y) in G_2 with $|N'| = k_2 - 1$ and $x \in X$ and $y \in Y$. We claim that N' contains the contracted vertex pq for otherwise N' would be a cutset of size $k_2 - 1$ in G . Therefore, it follows that (X, N, Y) where $N = N' \cup \{p, q\} - \{pq\}$ is a vertex-tri-partition in G that separates x from y . Note that $|N| = k_2$ and N includes *both* p and q . For the latter reason we note that (X, N, Y) is a vertex-tri-partition also in G_1 .

Subsequently, we work with the two vertex tri-partitions (S, M, T) and (X, N, Y) in G_1 (we stress that we work in G_1 and not in G or G_2). Recall that $s, p \in S$, and $t, q \in T$, and that M has size $k_1 - 1$; also, N separates x from y , and $p, q \in N$. Fig. 1 (a) below shows these vertex tri-partitions. Since M and N contain only white vertices, all terminals are in S or T , and in X or Y . We say that $S \cap X$ is *diagonally opposite* from $T \cap Y$, and $S \cap Y$ is diagonally opposite from $T \cap X$. Let A, B, C, D denote $S \cap N, X \cap M, T \cap N$ and $Y \cap M$ respectively, with I denoting $N \cap M$; note that A, B, C, D, I partition $M \cup N$.

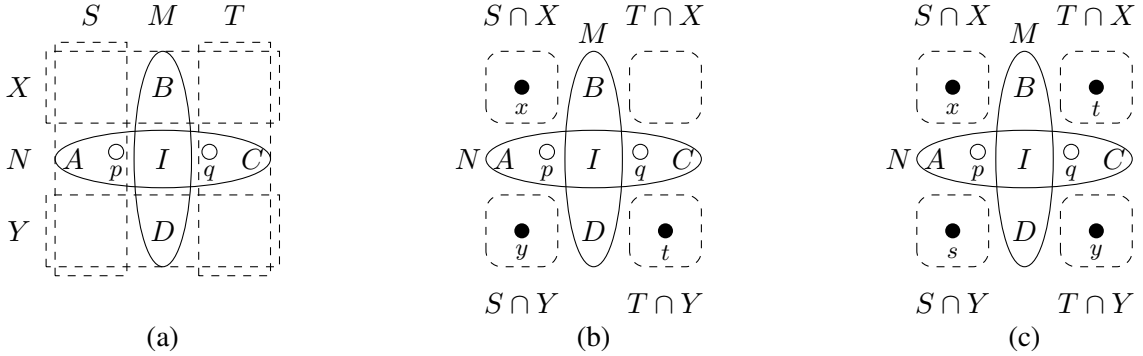


Figure 1: Part (a) illustrates the vertex tri-partitions (S, M, T) and (X, N, Y) .

In parts (b) and (c), we consider possible locations of the terminals s, t, x, y .

We assume w.l.o.g. that $x \in S$. If we also have $y \in S$, then $x \in S \cap X$ and $y \in S \cap Y$; therefore, one of x, y is diagonally opposite from t , suppose this is x . Fig. 1 (b) illustrates this case. Observe that $A \cup I \cup B$ separates x from y ; since x and y are k_2 -connected and $|N| = A \cup I \cup C = k_2$, it follows that $|B| \geq |C|$. Similarly, $C \cup I \cup D$ separates t from s , and since C contains q , Fact 1 implies that $|C \cup I \cup D| \geq k_1 > |B \cup I \cup D| = |M| = k_1 - 1$. Therefore, $|C| > |B|$, and we have a contradiction.

Hence, it must be that $y \notin S$; so $y \in T \cap Y$. The argument above shows that x and t cannot be diagonally opposite, so t must be in $T \cap X$. Similarly, s and y cannot be diagonally opposite, so $s \in S \cap Y$. Fig. 1 (c) shows the required positions of the vertices. Now, N separates s from t and contains p, q ; therefore, from fact 1, $|N| \geq k_1 > |M|$. But M separates x from y , and fact 2 implies that x, y are k_2 -connected in G_1 ; therefore, $|M| \geq k_2 = |N|$, and we have a contradiction. \square

3 Packing Element-Disjoint Steiner Trees and Forests

Consider a graph $G(V, E)$, with its vertex set V partitioned into T_1, T_2, \dots, T_m, W . We refer to each T_i as a group of *terminals*, and W as the set of Steiner or white vertices; we use $T = \bigcup_i T_i$ to denote the set of all terminals. A Steiner Forest for this graph is a forest that is a subgraph of G , such that each T_i is entirely contained in a single tree of this forest. (Note that T_i and T_j can be in the same tree.) For any group T_i of terminals, we define $\kappa'(T_i)$, the element-connectivity of T_i , as the largest k such that for every $u, v \in T_i$, the element-connectivity of u and v in the graph G is at least k .

We say two Steiner Forests for G are element-disjoint if they share no edges or Steiner vertices. (Every Steiner Forest must contain all the terminals.) The Steiner Forest packing problem is to find as many element-disjoint Steiner Forests for G as possible. By inserting a Steiner vertex between any pair of adjacent terminals,

we can assume that there are no edges between terminals, and then the problem of finding element-disjoint Steiner forests is simply that of finding Steiner forests that do not share any Steiner vertices. A special case is when $m = 1$ in which case we seek a maximum number of element-disjoint Steiner trees.

Proposition 3.1. *If $k = \min_i \kappa'_G(T_i)$, there are at most k element-disjoint Steiner Forests in G .*

Cheriy and Salavatipour [9] proved that if there is a single group T of terminals, with $\kappa'(T) = k$, then there always exist $\Omega(k/\log |T|)$ Steiner trees. Their algorithm proceeds by using Theorem 1.3, the global element-connectivity reduction of [21], to delete and contract edges between Steiner vertices, while preserving $\kappa'(T) = k$. Then, once we obtain a bipartite graph G' with terminals on one side and Steiner vertices on the other side, randomly color the Steiner vertices using $k/6 \log |T|$ colors; they show that w.h.p., each color class connects the terminal set T , giving $k/6 \log |T|$ trees. The bipartite case can be cast as a special case of packing bases of a polymatroid and a variant of the random coloring idea is applicable in this more general setting [3]; a derandomization is also provided in [3], thus yielding a deterministic polynomial time algorithm to find $\Omega(k/\log |T|)$ element-disjoint Steiner trees.

In this section, we give algorithms for packing element-disjoint Steiner Forests, where we are given m groups of terminals T_1, T_2, \dots, T_m . The approach of [9] encounters two difficulties. First, we cannot reduce to a bipartite instance, using only the global-connectivity version of the Reduction Lemma. In fact, our strengthening of the Reduction Lemma to preserve local connectivity was motivated by this; using it allows us once again assume that we have a bipartite graph $G'(T \cup W, E)$. Second, we cannot apply the random coloring algorithm on the bipartite graph G' directly; we give an example in Appendix A to show that this approach does not work. One reason for this is that, unlike the Steiner tree case, it is no longer a problem of packing bases of a submodular function. To overcome this second difficulty we use a decomposition technique followed by the random coloring algorithm to prove that there always exist $\Omega(k/(\log |T| \log m))$ element-disjoint forests. We believe that the bound can be improved to $\Omega(k/\log |T|)$.

We also consider the packing problem in restricted classes of graphs, in particular planar graphs. We obtain a much stronger bound, showing the existence of $\lceil k/5 \rceil - 1$ Steiner forests. The (simple) technique extends to graphs of fixed genus to prove the existence of $\Omega(k)$ Steiner forests where the constant depends mildly on the genus. We believe that there exist $\Omega(k)$ Steiner forests in any H -minor-free graph where H is fixed; it is shown in [1] that there exist $\Omega(k)$ Steiner *trees* in H -minor-free graphs. Our technique for planar graphs does not extend directly, but generalizing this technique allows us to make partial progress; by using our general graph result and some related ideas, in Section 3.3, we prove that in graphs of any fixed treewidth, there exist $\Omega(k)$ element-disjoint Steiner Trees if the terminal set is k -element-connected.

3.1 An $O(\log |T| \log m)$ -approximation for Packing in General Graphs

In order to pack element-disjoint Steiner forests we borrow the basic idea from [6] in the *edge-connectivity* setting for Eulerian graphs; this idea was later used by Lau [33] in the much more difficult non-Eulerian case. The idea at a high level is as follows: If all the terminals are k -connected then we can treat the terminals as forming one group and reduce the problem to that of packing Steiner trees. Otherwise, we can find a cut $(S, V \setminus S)$ that separates some groups from others. If the cut is chosen appropriately we may be able to treat one side, say S , as containing a single group of terminals and pack Steiner *trees* in them *without* using the edges crossing the cut. Then we can shrink S and find Steiner forests in the reduced graph; unshrinking of S is possible since we have many trees on S . In [6, 33] this scheme works to give $\Omega(k)$ edge-disjoint Steiner forests. However, the approach relies strongly on properties of edge-connectivity as well as the properties of the packing algorithm for Steiner trees. These do not generalize easily for element-connectivity. Nevertheless, we show that the basic idea can be applied in a slightly weaker way (resulting in the loss of an $O(\log m)$ factor over the Steiner tree packing factor). We remark that the reduction to a bipartite instance using the Reduction Lemma plays a critical role. A key definition is the notion of a good separator given below.

Definition 3.2. Given an graph $G(V, E)$ with terminal sets T_1, T_2, \dots, T_m , such that for all i , $\kappa'(T_i) \geq k$, we say that a set S of white vertices is a good separator if (i) $|S| \leq k/2$ and (ii) there is a component of $G - S$ in which all terminals are $k/2 \log m$ -element-connected.

Note that the empty set is a good separator if all terminals are $k/2 \log m$ -element-connected.

Lemma 3.3. For any instance of the Steiner Forest Packing problem, there is a polynomial-time algorithm that finds a good separator.

Proof: Let $G(V, E)$ be an instance of the Steiner Forest packing problem, with terminal sets T_1, T_2, \dots, T_m such that each T_i is k -element-connected. If T is $\frac{k}{2 \log m}$ -element connected, the empty set S is a good separator.

Otherwise, there is some set of white vertices of size less than $\frac{k}{2 \log m}$ that separates some of the terminals from others. Let S_1 be a minimal such set, and consider the two or more components of $G - S_1$. Note that each T_i is entirely contained in a single component, since T_i is at least k -element-connected, and $|S_1| < k$. Among the components of $G - S_1$ that contain terminals, consider a component G_1 with the fewest sets of terminals; G_1 must have at most $m/2$ sets from T_1, \dots, T_m . If the set of all terminals in G_1 is $\frac{k}{2 \log m}$ connected, we stop, otherwise, find in G_1 a set of white vertices S_2 with size less than $\frac{k}{2 \log m}$ that separates terminals of G_1 . Again, find a component G_2 of $G_1 - S_2$ with fewest sets of terminals, and repeat this procedure until we obtain some subgraph G_ℓ in which all the terminals are $\frac{k}{2 \log m}$ -connected. We can always find such a subgraph, since the number of sets of terminals is decreasing by a factor of 2 or more at each stage, so we find at most $\log m$ separating sets S_j . Now, we observe that the set $S = \bigcup_{j=1}^{\ell} S_j$ is a good separator. It separates the terminals in G_ℓ from the rest of T , and its size is at most $\log m \times \frac{k}{2 \log m} = k/2$; it follows that each set of terminals T_i is entirely within G_ℓ , or entirely outside it. By construction, all terminals in G_ℓ are $\frac{k}{2 \log m}$ connected. \square

We can now prove our main result, that we can always find a packing of $\Omega(\frac{k}{\log |T| \log m})$ Steiner forests.

Theorem 3.4. Given a graph $G(V, E)$, with terminal sets T_1, T_2, \dots, T_m , such that for all i , $\kappa'(T_i) \geq k$, there is a polynomial-time algorithm to pack $\Omega(k / \log |T| \log m)$ element-disjoint Steiner Forests in G .

Proof: The proof is by induction on m . The base case of $m = 1$, follows from [9, 3]; G contains at least $\frac{k}{6 \log |T|}$ element-disjoint Steiner Trees, and we are done.

We may assume G is bipartite by using the Reduction Lemma. Find a good separator S , and a component G_ℓ of $G - S$ in which all terminals are $\frac{k}{2 \log m}$ -connected. Now, since the terminals in G_ℓ are $\frac{k}{2 \log m}$ -connected, use the algorithm of [9] to find $\frac{k}{12 \log m \log |T|}$ element-disjoint Steiner trees containing all the terminals in G_ℓ ; none of these trees uses vertices of S . Number these trees from 1 to $\frac{k}{12 \log m \log |T|}$; let \mathcal{T}_j denote the j th tree.

The set S separates G_ℓ from the terminals in $G - G_\ell$. If S is not a minimal such set, discard vertices until it is. If we delete G_ℓ from G , and add a clique between the white vertices in S to form a new graph G' , it is clear that the element-connectivity between any pair of terminals in G' is at least the element-connectivity they had in G . The graph G' has $m' \leq m - 1$ groups of terminals; by induction, we can find $\frac{k}{12 \log |T| \log m} < \frac{k}{12 \log |T| \log m'}$ element-disjoint Steiner forests for the terminals in G' . As before, number the forests from 1 to $\frac{k}{12 \log m \log |T|}$; we use \mathcal{F}_j to refer to the j th forest. These Steiner Forests may use the newly added edges between the vertices of S ; these edges do not exist in G . However, we claim that the Steiner Forest \mathcal{F}_j of G' , together with the Steiner tree \mathcal{T}_j in G_ℓ gives a Steiner Forest of G . The only way this might not be true is if \mathcal{F}_j uses some edge added between vertices $u, v \in S$. However, every vertex in S is adjacent to a terminal in G_ℓ , and all the terminals of G_ℓ are in every one of the Steiner trees we generated. Therefore, there is a path from u to v in \mathcal{T}_j . Hence, deleting the edge between u and v from \mathcal{F}_j still leaves each component of $\mathcal{F}_j \cup \mathcal{T}_j$ connected.

Therefore, for each $1 \leq j \leq \frac{k}{12 \log m \log |T|}$, the vertices in $\mathcal{F}_j \cup \mathcal{T}_j$ induce a Steiner Forest for G . \square

3.2 Packing Steiner Trees and Forests in Planar Graphs

We now prove much improved results for restricted classes of graphs, in particular planar graphs. If G is planar, we show the existence of $\lceil k/5 \rceil - 1$ element-disjoint Steiner Forests.¹ The intuition and algorithm are easier to describe for the Steiner tree packing problem and we do this first. We achieve the improved bound by observing that planarity restricts the use of many white vertices as “branch points” (that is, vertices of degree ≥ 3) in forests. Intuitively, even in the case of packing trees, if there are terminals t_1, t_2, t_3, \dots that must be in every tree, and white vertices $w_1, w_2, w_3 \dots$ that all have degree 3, it is difficult to avoid a $K_{3,3}$ minor. Note, however, that degree 2 white vertices behave like edges and do not form an obstruction. We capture this intuition more precisely by showing that there must be a pair of terminals t_1, t_2 that are connected by $\Omega(k)$ degree-2 white vertices; we can contract these “parallel edges”, and recurse.

We describe below an algorithm for packing Steiner Trees. Through the rest of the section, we assume $k > 10$; otherwise, $\lceil k/5 \rceil - 1 \leq 1$, and we can always find 1 Steiner Tree in a connected graph.

Given an instance of the Steiner Tree packing problem in planar graphs, we construct a *reduced instance* as follows: Use the Reduction Lemma to delete and contract edges between white vertices to obtain a planar graph with vertex set $T \cup W$, such that W is a stable set. Now, for each vertex $w \in W$ of degree 2, connect the two terminals that are its endpoints directly with an edge, and delete w . (All edges have unit capacity.) We now have a planar *multigraph*, though the only parallel edges are between terminals, as these were the only edges added while deleting degree-2 vertices in W . Note that this reduction preserves the element-connectivity of each pair of terminals; further, any set of element-disjoint trees in this reduced instance corresponds to a set of element-disjoint trees in the original instance. We need the following technical result:

Theorem 3.5 (Borodin, [2]). *If G is a planar graph with minimum degree 3, it has an edge of weight at most 13, where the weight of an edge is the sum of the degrees of its endpoints.*

Lemma 3.6. *In a reduced instance of the Planar Steiner Tree Packing problem, if T is k -element-connected, there are two terminals t_1, t_2 with at least $\lceil k/5 \rceil - 1$ parallel edges between them.*

Proof: We prove this lemma in Appendix A.1; here, we give a proof showing the weaker result that there exist terminals t_1, t_2 with $\lceil k/10 \rceil$ edges between them. Let G be the planar multigraph of the reduced instance. Since T is k -element-connected in G , every terminal has degree at least k in G . Construct a planar graph G' from G by keeping only a single copy of each edge. We argue below that some terminal $t_1 \in T$ has degree at most 10 in G' ; it follows that G must contain at least $\lceil k/10 \rceil$ copies of some edge incident to t_1 , as t_1 has degree at least k in G . These edges must be incident to another terminal t_2 , completing the proof.

To see that some terminal t_1 has degree at most 10 in G' , we first assume that no terminal has degree ≤ 2 , or we are already done. Now, as every vertex of W in a reduced instance has degree at least 3, we may use Theorem 3.5; this implies that G' has an edge e , such that the sum of the degrees of the endpoints of e is at most 13. The edge e must be incident to a terminal t_1 , as the white vertices are a stable set. The other endpoint of e has degree at least 3, so the degree of t_1 is at most 10. \square

It is now easy to prove by induction that we can pack $\lceil k/5 \rceil - 1$ disjoint trees.

Theorem 3.7. *Given an instance of the Steiner Tree packing problem on a planar graph G with terminal set T , if $\kappa'(T) \geq k$, there is a polynomial-time algorithm to find at least $\lceil k/5 \rceil - 1$ element-disjoint Steiner trees in G . Moreover, in each tree, the white (non-terminal) vertices all have degree 2.*

Proof: We prove this theorem by induction on $|T|$; if $|T| = 2$, there are k disjoint *paths* in G from one terminal to the other, so we are done (including the guarantee of degree 2 for white vertices).

¹Note that in the special case of packing Steiner Trees, the paper of Aazami *et al.* [1] shows that there are $\lfloor k/2 \rfloor - 1$ element-disjoint Steiner Trees.

Otherwise, apply the Reduction Lemma to construct a reduced instance G' , preserving the element-connectivity of T . Now, from Lemma 3.6, there exist a pair of terminals t_1, t_2 that have $\lceil k/5 \rceil - 1$ parallel edges between them (Note that the parallel edges between t_1 and t_2 may have non-terminals on them in the original graph but they have degree 2.). Contract t_1, t_2 into a single terminal t , and consider the new instance of the Steiner Tree packing problem with terminal set $T' = T \cup \{t\} - \{t_1, t_2\}$. It is easy to see that the element-connectivity of the terminal set is still at least k ; by induction, we can find $\lceil k/5 \rceil - 1$ Steiner trees containing all the terminals of T' , with the property that all non-terminals have degree 2. Taking these trees together with $\lceil k/5 \rceil - 1$ edges between t_1 and t_2 gives $\lceil k/5 \rceil - 1$ trees in G' that span the original terminal set T . \square

Packing Steiner Forests in Planar Graphs: The algorithm described above for packing Steiner trees encounters a technical difficulty when we try to extend it to Steiner forests. Lemma 3.6 can be used at the start to merge some two terminals. However, as the algorithm proceeds it may get stuck in the following situation: it merges all terminals from some group T_i into a single terminal. Now this terminal does not require any more connectivity to other terminals although other groups are not yet merged together. In this case we term this terminal as dead. In the presence of dead terminals Lemma 3.6 no longer applies; we illustrate this with a concrete example in Appendix A.2. We overcome this difficulty by showing that a dead terminal may be replaced by a grid of white vertices — the grid is necessary to ensure that the resulting graph is still planar. We can then apply the Reduction Lemma to remove edges between the newly added white vertices and proceed with the merging process. See Appendix A.2 for details.

Extensions: Our result for planar graphs can be generalized to graphs of fixed genus; Ivanco [22] generalized Theorem 3.5 to show that a graph G of genus g has an edge of weight at most $2g + 13$ if $0 \leq g \leq 3$ and an edge of weight at most $4g + 7$ otherwise. This allows us to prove that there exist $\lceil k/c \rceil$ forests where $c \leq 4g + 8$; we have not attempted to optimize this constant c . Aazami *et al.* [1] also give algorithms for packing Steiner Trees in these graph classes, and graphs excluding a fixed minor. We thus make the following natural conjecture:

Conjecture 1. *Let $G = (V, E)$ be a H -minor-free graph, with terminal sets T_1, T_2, \dots, T_m , such that for all i , $\kappa'(T_i) \geq k$. There exist $\Omega(k/c)$ element-disjoint Steiner forests in G , where c depends only on the size of H .*

We note that Lemma 3.6 fails to hold for H -minor-free graphs, and in fact fails even for bounded treewidth graphs. Thus, our approach cannot be directly generalized. However, instead of attempting to contract together just two terminals connected by many parallel edges, we may be able contract together a constant number of terminals that are “internally” highly connected. Using Theorem 3.4 and other ideas, we prove in the next section that this approach suffices to pack many trees in graphs with small treewidth. We believe that these ideas together with the structural characterization of H -minor-free graphs by Robertson and Seymour [37] should lead to a positive resolution of Conjecture 1.

3.3 Packing Trees in Graphs of Bounded Treewidth

Let $G(V, E)$ be a graph of treewidth $\leq r - 1$, with terminal set $T \subseteq V$ such that $\kappa'(T) \geq k$. In this section, we give an algorithm to find, for any fixed r , $\Omega(k)$ element-disjoint Steiner Trees in G . Our approach is similar to that for packing Steiner Trees in planar graphs, where we argued in Lemma 3.6 that there exist two terminals t_1, t_2 with $\Omega(k)$ parallel edges between them, so we could contract them together and recurse on a smaller instance. In graphs of bounded treewidth, this is no longer the case; see the end of Appendix A for an example in which no pair of terminals is connected by many parallel edges. However, we argue that there exists a small set of terminals $T' \subset T$ that is highly “internally connected”, so we can find $\Omega(k)$ disjoint trees connecting all terminals in T' , without affecting the connectivity of terminals in $T - T'$. We can then contract together T' and the white vertices used in these trees to form a single new terminal t , and again recurse on a smaller instance. The following lemma captures this intuition:

Lemma 3.8. *If $G(V, E)$ is a bipartite graph of treewidth at most $r - 1$, with terminal set $T \subset V$ such that $|T| \geq 2^r$, $\kappa'(T) \geq k$, there exists a set $S \subseteq V - T$ such that there is a component G' of $G - S$ containing $k/12r^2 \log(3r)$ element-disjoint Steiner trees for the (at least 2) terminals in G' . Moreover, these trees in G' can be found in polynomial time.*

Given this lemma, we prove below that for any fixed r , we can pack $\Omega(k)$ element-disjoint trees in graphs of treewidth at most $r - 1$. The proof combines ideas of Theorem 3.7 and Theorem 3.4.

Theorem 3.9. *Let $G = (V, E)$ be a graph of treewidth at most $r - 1$. For any terminal set $T \subseteq V$ with $\kappa'_G(T) \geq k$, there exist $\Omega(k/12r^2 \log(3r))$ element-disjoint Steiner trees on T .*

Proof: As for Theorem 3.7, we prove this theorem by induction. Let G be a graph of treewidth at most $r - 1$, with terminal set T . If $|T| \leq 2^r$, we have $k/6 \log |T| \geq k/6r$ element-disjoint trees from the tree-packing algorithm of Cheriyan and Salavatipour [9] in arbitrary graphs.

Otherwise, we use the Reduction Lemma to ensure that G is bipartite. Let S be a set of white vertices guaranteed to exist from Lemma 3.8. If S is not a minimal such set, discard vertices until it is. Now, find $k/12r^2 \log(3r)$ element-disjoint trees containing all terminals in some component G' of $G - S$; note that each vertex of S is incident to some terminal in G' , and hence to every tree. (This follows from the minimality of S and the fact that G is bipartite.) Modify G by contracting all of G' to a single terminal t , and make it incident to every vertex of S . It is easy to see that all terminals in the new graph are k -element-connected; therefore, we now have an instance of the Steiner Tree packing problem on a graph with fewer terminals. The new graph has treewidth at most $r - 1$, so by induction, we have $k/12r^2 \log(3r)$ element-disjoint trees for the terminals in this new graph; taking these trees together with the $k/12r^2 \log(3r)$ trees of G' gives $k/12r^2 \log(3r)$ trees of the original graph G . \square

We devote the rest of this section to proving the crucial Lemma 3.8. Subsequently, we may assume, w.l.o.g. (after using the Reduction Lemma) that the graph G is bipartite; we may further assume that $k \geq 12r^2 \log(3r)$ and $|T| \geq 2^r$. First, observe that G has a small cutset that separates a few terminals from the rest.

Proposition 3.10. *G has a cutset C of size at most r such that some component of $G - C$ contains between r and $2r$ terminals.*

Proof Sketch: Fix a tree-decomposition \mathcal{T} of G ; every non-leaf node of \mathcal{T} corresponds to a cutset, and each node of \mathcal{T} contains at most r vertices of G . Start at a leaf of \mathcal{T} , and walk upwards until reaching a node v such that the subtree of \mathcal{T} rooted at some child of v contains between r and $2r$ terminals. (This is always possible since walking up one step only gives at most r more terminals.) \square

We find the set S and component of $G - S$ in which we contract together a small number of terminals by focusing on the cutset C and component of $G - C$ that are guaranteed to exist from the previous proposition. We introduce some notation before proceeding with the proof:

1. Let C be a cutset of size at most r , and let V' be the vertices of a component of $G - C$ containing between r and $2r$ terminals.
2. Since terminals in V' are k -connected to the terminals in the rest of the graph, and $|C| \leq r \ll k$, C contains at least one black vertex. Let C' be the set of black vertices in C .
3. Let $G' = G[V' \cup C']$ be the graph induced by V' and C' .

We omit a proof of the following straightforward proposition; the second part of the statement follows from the fact that each terminal in V' is k -connected to terminals outside G' , and these paths to terminals outside G' must go through the cutset C' of size at most r .

Proposition 3.11. *The graph G' contains between r and $3r$ terminals (as C' may contain up to r terminals), and each terminal in V' is at least k/r -connected to some terminal in C' .*

Let T' be the set of terminals in G' . If $k'_{G'}(T') \geq k/2r^2$, we can easily find a set of white vertices satisfying Lemma 3.8: Let S be the set of vertices of G that are adjacent (in G) to vertices of G' . It is obvious that S separates G' from the rest of G , and all terminals in T' are highly connected; from the tree packing result of [9], we can find the desired disjoint trees in G' . Finally, note that all vertices of S are white, as the only neighbors of G' are either white vertices of the cutset C or the neighbors of the black vertices in C , all of which are white as G is bipartite.

However, it may not be the case that all terminals of T' are highly connected in G' . In this event, we use the following simple algorithm (very similar to that in the proof of Lemma 3.3) to find a highly-connected subset of T' : Begin by finding a set S_1 of at most $k/2r^2$ white vertices in G' that separates terminals of T' . Among the components of $G' - S_1$, pick a component G_1 with at least one terminal of V' . If all terminals of G_1 are $k/2r^2$ connected, stop; otherwise, find in G_1 a set S_2 of at most $k/2r^2$ white vertices that separates terminals of G_1 , pick a component G_2 of $G_1 - S_2$ that contains at least one terminal of V' , and proceed in this manner until finding a component G_ℓ in which all terminals are $k/2r^2$ connected.

Claim 3.12. *We perform at most r iterations of this procedure before we stop, having found some subgraph G_ℓ in which all the (at least 2) terminals are $k/2r^2$ connected.*

Proof: At least one terminal of C' must be lost every time we find such a set S_i ; if this is true, the claim follows. To see that this is true, observe that when we find a cutset S_{i+1} in G_i , there is a component that we do *not* pick that contains a terminal t . If this terminal t is in C' , we are done; otherwise, it must be in V' . But from Proposition 3.11 all terminals in V' are k/r connected to some terminal in C' , and so some terminal of C' must be in the same component as t . When we stop with the subgraph G_ℓ , it contains at least one terminal $t' \in V'$, and at least one terminal of C' to which t' is highly connected; therefore, G_ℓ contains at least 2 terminals. \square

All terminals in the subgraph G_ℓ are $k/2r^2$ -connected, and there are at most $3r$ of them, so we can find $k/12r^2 \log(3r)$ disjoint trees in G_ℓ that connect them, using the tree-packing result of [9]. Let S be the set of vertices of G that are adjacent (in G) to vertices of G_ℓ ; obviously, S separates G_ℓ from the rest of G , and to satisfy Lemma 3.8, it merely remains to verify that S only contains white vertices. Every terminal in $G' - G_\ell$ was separated from G_ℓ by white vertices in some S_i , and terminals in $G - G'$ can only be incident to white vertices of the cutset C , which are not in G' , let alone G_ℓ . This completes the proof of Lemma 3.8.

4 Single-Sink Vertex-Connectivity

Recall that in the SS- k -CONNECTIVITY problem, one is given an undirected graph $G = (V, E)$ with edge costs, a specified sink/root vertex r , and a subset of terminals $T \subseteq V$, with $|T| = h$. The goal is to find a minimum cost subgraph H that contains k vertex-disjoint paths from each terminal $t \in T$ to the root. In this section we give a very simple proof of the main technical result in [12] using the Reduction Lemma. We lead up to the technical lemma via a description of the (simple) algorithm for SS- k -CONNECTIVITY.

The basic algorithmic idea comes from [4]; this is the idea of using *augmentation*. Let $T' \subseteq T$ be a subset of terminals and let H' be a subgraph of G that is feasible for T' . For a terminal $t \in T \setminus T'$, a set of k paths p_1, \dots, p_k is said to be an augmentation for t with respect to T' if (i) p_i is a path from t to some vertex in $T' \cup \{r\}$ (ii) the paths are internally vertex disjoint and (iii) a terminal $t' \in T'$ is the endpoint of at most one of the k paths. Note that the root is allowed to be the endpoint of more than one path. The following proposition is easy to prove via a simple min-cut argument.

Proposition 4.1. *If p_1, p_2, \dots, p_k is an augmentation for t with respect to T' and H' is a feasible solution for the SS- k -CONNECTIVITY instance with terminal set T' , then $H \cup (\bigcup_i p_i)$ is a feasible solution for $T' \cup \{t\}$.*

Given T' and t , the augmentation cost of t with respect to T' is the cost of a min-cost set of paths that augment t w.r.t. to T' . We can find the augmentation cost for a terminal t by solving a simple min-cost flow problem. The key theorem in [12] is the following.

Theorem 4.2 (Vertex-Connectivity, [12]). *If OPT denotes the cost of an optimal solution to SS- k -CONNECTIVITY, and $\text{AugCost}(t)$ the cost of an augmentation for terminal t w.r.t. $T - \{t\}$, $\sum_t \text{AugCost}(t) \leq 8k \cdot \text{OPT}$.*

We now briefly describe the algorithm of [5] for SS- k -CONNECTIVITY; a variant is used in [4, 12].

Permute the terminals randomly; let t_j denote the j th terminal in the permutation and let $T_j = \{t_1, \dots, t_j\}$.
 Subgraph $H \leftarrow \emptyset$
 For $i = 1$ to $|T|$.
 Add to H a min-cost augmentation of t_i with respect to T_{i-1} .
 Output the subgraph H .

Note that the above is a greedy algorithm except for the initial randomization. Interestingly, as noted in [5], the randomization is key; even for $k = 2$ there exist permutations that yield a solution of cost $\Omega(|T| \cdot \text{OPT})$. Using Theorem 4.2 it is easy to prove that the above algorithm is a randomized $O(k \log |T|)$ -approximation for SS- k -CONNECTIVITY: simply observe that the *expected* augmentation cost for the last terminal in the permutation is at most $8k\text{OPT}/|T|$; a straightforward inductive argument then completes the proof.

The main ingredient in the proof of Theorem 4.2, as shown by [12], is the following weaker statement involving paths that are *element-disjoint*, as opposed to vertex-disjoint.

Lemma 4.3 (Element-Connectivity, [12]). *Given an instance of SS- k -CONNECTIVITY, let $\text{ElemCost}(t)$ denote the minimum cost of a set of k internally vertex-disjoint paths from any terminal t to $T \cup \{r\} - t$. Then, $\sum_{t \in T} \text{ElemCost}(t) \leq 2\text{OPT}$, where OPT is the cost of an optimal solution to this instance.*

It is shown in [12] that one can prove Theorem 4.2 by repeatedly invoking Lemma 4.3 to obtain a large collection of paths from each $t \in T$ to other terminals, and applying a flow-scaling argument. The heart of the proof of the crucial Lemma 4.3, is a structural theorem of [12] on *spiders*: A spider is a tree containing at most a single vertex of degree greater than 2. If such a vertex exists, it is referred to as the *head* of the spider, and each leaf is referred to as a *foot*. Thus, a spider may be viewed as a collection of disjoint paths (called *legs*) from its feet to its head. If the spider has no vertex of degree 3 or more, any vertex of the spider may be considered its head. Vertices that are not the head or feet are called intermediate vertices of the spider. The Reduction Lemma allows us to give an extremely easy inductive proof of the Spider Decomposition Theorem below,² greatly simplifying the proof of [12].

Theorem 4.4 ([12]). *Let $G(V, E)$ be a graph with a set $B \subseteq V$ of black vertices such that every pair of black vertices is k -element connected. There is a subgraph H of G whose edges can be partitioned into spiders such that:*

1. *For each spider, its feet are distinct black vertices, and all intermediate vertices are white.*
2. *Each black vertex is a foot of exactly k spiders, and each white vertex appears in at most one spider.*
3. *If a white vertex is the head of a spider, the spider has at least two feet.*

Before giving the formal short proof we remark that if the graph is bipartite then the collection of spiders is trivial to see: they are simply the edges between the black vertices and the stars rooted at each white vertex!

²In the decomposition theorem of [12], the spiders satisfy a certain additional technical condition; the proof of Theorem 4.2 in [12] relies on this condition. We give a modified proof of Theorem 4.2 that does not require the condition.

Thus the Reduction Lemma effectively allows us to reduce the problem to a trivial case.

Proof: We prove this theorem by induction on the number of edges between white vertices in G . As the base case, we have a graph G with no edges between white vertices; therefore, G is bipartite. (Recall that there are no edges between black vertices.) Each pair of black vertices is k -element connected, and hence every black vertex has at least k white neighbors. Let every $b \in B$ mark k of its (white) neighbors arbitrarily. Every white vertex w that is marked at least twice becomes the head of a spider, the feet of which are the black vertices that marked w . For each white vertex w marked only once, let b be its neighbor that marked it, and b' be another neighbor. We let $b - w - b'$ be a spider with foot b and head b' . It is easy to see that the spiders are disjoint, and that they satisfy all the other desired conditions.

For the inductive step, consider a graph G with an edge pq between white vertices. If all black vertices are k -element connected in $G_1 = G - pq$, then we can apply induction, and find the desired subgraph of G_1 and hence of G . Otherwise, by Theorem 1, we can find the desired set of spiders in $G_2 = G/pq$. If the new vertex $v = pq$ is not in any spider, this set of spiders exists in G , and we are done. Otherwise, let S be the spider containing v . If v is not the head of S , let x, y be its neighbors in S . Either x and y are both adjacent to p , or both adjacent to q , or (w.l.o.g.) x is adjacent to p and y to q . Therefore, we can replace the path $x - v - y$ in S with one of $x - p - y$, $x - q - y$, or $x - p - q - y$. If v is the head of S , we know that it has at least 2 feet. If at least 2 legs of S are incident to each of p and q , we can create two new spiders S_p and S_q , with heads p and q respectively; S_p contains the legs of S incident to p , and S_q the legs incident to q . If all the legs of S are incident to p , we let p be the head of the spider in G ; the case in which all legs are incident to q is symmetric. If neither of these cases holds, it follows that (w.l.o.g.) exactly one leg ℓ of S is incident to p , with the remaining legs being incident to q . We let q be the head of the new spider, and add p to the leg ℓ . \square

The authors of [12] showed that, once we have the Spider Decomposition Theorem, it is very easy to prove Lemma 4.3.

Proof of Lemma 4.3:([12]) In an optimal solution H to an instance of SS- k -CONNECTIVITY, every terminal is k -vertex-connected to the root. Let the terminals be black vertices, and non-terminals be white; it follows that all the terminals are k -element connected to the root in H , and hence to each other. Therefore, we can find a subgraph of H of total cost at most OPT which can be partitioned into spiders as in Theorem 4.4. For each spider S and every terminal t that is a foot of S , we find a path entirely contained within S from t to another terminal. Each edge of S is in at most two such paths; since the spiders are disjoint and each terminal is a foot of k spiders, we obtain the desired result.

If the head of S is a terminal, the path for each foot is simply the leg of S from that foot to the head. Each edge of S is in a single path. If the head of S is a white vertex, it has at least two feet. Fix an arbitrary ordering of the feet of S ; the path for foot i follows leg i from the foot to the head, and then leg $i + 1$ from the head to foot $i + 1$. (The path for the last foot follows the last leg, and then leg 1 from the head to the foot.) It is easy to see that each edge of S is in exactly two paths; this completes the proof. \square

Finally, we give a proof of Theorem 4.2 that relies only on the statement of Lemma 4.3. Our proof is a technical modification of the one in [12] and as previously remarked, does not need rely on the additional condition on the spiders that [12] guarantees. Our proof also gives a slightly stronger bound on $\sum_t AugCost(t)$ ($8k \cdot OPT$ instead of $(18k + 3) \cdot OPT$).

Proof of Theorem 4.2: We give an algorithm to find an augmentation for each terminal that proceeds in $4k^2$ iterations: In each iteration, for every terminal t , it finds a set of k internally vertex-disjoint paths from t to other terminals or the root. Let $\mathcal{P}_i(t)$ denote the set of paths found for terminal t in iteration i . These paths have the following properties:

1. For each terminal t , every other terminal is an end-point of fewer than $4k^2 + 2k$ paths in $\bigcup_i \mathcal{P}_i(t)$.
2. In each iteration i , $\sum_t Cost(\mathcal{P}_i(t)) \leq 4kOPT$.

Given these two properties, we can prove the theorem as follows: Separately for each terminal t , send 1 unit of flow along each of the paths in $\bigcup_i \mathcal{P}_i(t)$; we thus have a flow of $4k^2 \cdot k$ units from t to other terminals. Scale this flow down by $4k^2 \cdot (k + \frac{1}{2})/k$, to obtain a flow of $\frac{k^2}{k+1/2} > k - 1/2$ from t to other terminals. After the scaling step, the net flow through any vertex (terminal or non-terminal) is at most 1, since the maximum flow through a vertex before scaling was $4k^2 + 2k$. Let $FlowCost(t)$ denote the cost of this scaled flow for terminal t ; if we now scale the flow *up* by a factor of 2, we obtain a flow of value greater than $2k - 1$ from t to other terminals, in which the flow through any vertex besides t is at most 2. Therefore, by the integrality of min-cost flow, we can find an integral flow of $2k - 1$ units from t to other terminals, of total cost at most $2FlowCost(t)$. Let E_t be the set of edges used in this integral flow; it follows that $cost(E_t) \leq 2FlowCost(t)$. It is also easy to see that E_t contains k disjoint paths from t to k distinct terminals, by observing that a hypothetical cutset of size $k - 1$ contradicts the existence of the flow of value $2k - 1$ in which the flow through a vertex is at most 2.

Therefore, we have found k disjoint paths from t to k other terminals, of total cost $2FlowCost(t)$. To bound the cost over all terminals, we note that from the second property above, we have $\sum_t FlowCost(t) \leq 4k^2 \cdot 4kOPT / \left(4k^2 \frac{k+1/2}{k}\right)$, which is less than $4kOPT$. It follows that the total cost of the set of paths is at most $2 \sum_t FlowCost(t) < 8kOPT$.

It remains only to show that we can find a set of paths for each terminal in every iteration that satisfies the two desired properties. The proof below uses induction on the number of iterations i to prove property 1: After i iterations, for each terminal t , every other terminal is an end-point of fewer than $i + 2k$ paths in $\bigcup_i \mathcal{P}_i(t)$.

In iteration i , for each terminal t , let $Blocked(t)$ denote the set of terminals in $T - t$ that have been the endpoints of at least $(i - 1) + k$ paths in $\bigcup_{j=1}^{i-1} \mathcal{P}_j(t)$. (Note that the root r is never in any $Blocked(t)$.) Since the total number of paths that have been found so far is $(i - 1)k$, $|Blocked(t)| < k$. Construct a directed graph D on the set of terminals, with edges from each terminal t to the terminals in $Blocked(t)$. Since the out-degree of each vertex in D is at most $k - 1$, there is a vertex of in-degree at most $k - 1$; therefore, the digraph D is $2k - 2$ degenerate and so can be colored using $2k - 1$ colors. Let $C_1, C_2, \dots, C_{2k-1}$ denote the color classes in a proper coloring of D ; if $t_1, t_2 \in C_j$, then in iteration i , $t_1 \notin Blocked(t_2)$ and $t_2 \notin Blocked(t_1)$. For each color class C_j in turn, consider the terminals of C_j as black, and the non-terminals and terminals of other classes as white. There is a graph of cost OPT in which every terminal of C_j is k -vertex-connected to the root, so C_j is k -element-connected to the root in this graph even if terminals not in C_j are regarded as white vertices. From Lemma 4.3, for every C_j , we can find a set of internally disjoint paths from each $t \in C_j$ to $C_j \cup \{r\} - \{t\}$ of total cost at most $2OPT$. If these paths contain other terminals in $T - C_j$ as intermediate vertices, trim them at the first terminal they intersect. It follows that $\sum_j \sum_{t \in C_j} Cost(\mathcal{P}_i(t)) < 4kOPT$, establishing property 2 above.

To conclude, we show that for each terminal t , after iteration i , every other terminal is an end-point of fewer than $i + 2k$ paths in $\bigcup_{j=1}^i \mathcal{P}_j(t)$. Let C be the color class containing t ; if $t' \in Blocked(t)$, at most one new path in $\mathcal{P}_i(t)$ ends in t' , as the paths for t are disjoint except at terminals in C , and $t' \notin C$. By induction, before this iteration t' was the endpoint of fewer than $(i - 1) + 2k$ paths for t , and so after this iteration, it cannot be the endpoint of $i + 2k$ paths for t . If $t' \notin Blocked(t)$, it was the endpoint of at most $(i - 1) + k - 1$ paths for t before this iteration; even if all the k paths for t in this iteration ended at t' , it is the endpoint of at most $i + 2k - 2$ paths for t after the iteration. This gives us the desired property 1, completing the proof. \square

Theorem 4.2 and Lemma 4.3 have applications to more general problems including the node-weighted version of SS - k -CONNECTIVITY [12] and rent-or-buy and buy-at-bulk network design [5]. We omit discussion of these applications in this version of the paper.

5 Conclusions

Having generalized the reduction step of [21] to handle local element connectivity, we demonstrated applications of this stronger Reduction Lemma to packing element (and edge) disjoint Steiner trees and forests, and also to $SS-k$ -CONNECTIVITY. We believe that the Reduction Lemma will find other applications in the future. We close with several open questions:

- We believe that our bound on the number of element-disjoint Steiner forests in a general graph can be improved from $\Omega(k/(\log |T| \log m))$ to $\Omega(k/\log |T|)$.
- Prove or disprove Conjecture 1, on packing disjoint Steiner Forests in graphs excluding a fixed minor.
- In a natural generalization of the Steiner Forest packing problem, each non-terminal/white vertex has a *capacity*, and the goal is to pack forests subject to these capacity constraints. In general graphs, it is easy to reduce this problem to the uncapacitated/unit-capacity version (for example, by replacing a white vertex of capacity c by a clique of size c), but this is not necessarily the case for restricted classes of graphs. In particular, it would be interesting to pack $\Omega(k)$ forests for the capacitated planar Steiner Forest problem.
- The known hardness of approximation factor for $SS-k$ -CONNECTIVITY is $\Omega(\log n)$ when k is a polynomial function of n , the number of vertices [28]. Can the current ratio of $O(k \log |T|)$ be improved?

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References

- [1] A. Aazami, J. Cheriyan, and K.R. Jampani. Approximation Algorithms and Hardness Results for Packing Element-Disjoint Steiner Trees. Manuscript, 2008. Submitted to *Algorithmica*.
- [2] O.V. Borodin. Coupled Colorings of Graphs on a Plane. *Metody Diskret. Analiz.*, 45: 21–27, 1987. [In Russian]
- [3] G. Calinescu, C. Chekuri and J. Vondrak. Disjoint Bases in a Polymatroid. To appear in *Random Structures & Algorithms*.
- [4] T. Chakraborty, J. Chuzhoy and S. Khanna. Network Design for Vertex Connectivity. *Proc. of ACM STOC*, 167–176, 2008.
- [5] C. Chekuri and N. Korula. Single-Sink Network Design with Vertex-Connectivity Requirements. *Proc. of FST&TCS*, 2008.
- [6] C. Chekuri and B. Shepherd. Approximate Integer Decompositions for Undirected Network Design Problems. *SIAM J. on Disc. Math*, 23(1): 163–177, 2008.
- [7] J. Cheriyan. Personal Communication, October 2008.

- [8] J. Cheriyan and M. Salavatipour. Hardness and Approximation Results for Packing Steiner Trees. *Algorithmica*, 45(1): 21–43, 2006. Preliminary version in *Proc. of ESA*, 180–191, 2004.
- [9] J. Cheriyan and M. Salavatipour. Packing Element-disjoint Steiner Trees. *ACM Trans. on Algorithms*, 3(4), 2007. Preliminary version in *Proc. of APPROX 2005*, 52-61, 2005.
- [10] J. Cheriyan, S. Vempala and A. Vetta. An approximation algorithm for the minimum-cost k -vertex connected subgraph. *SIAM J. on Computing*, 32(4):1050–1055, 2003.
- [11] J. Cheriyan, S. Vempala and A. Vetta. Network design via iterative rounding of setpair relaxations. *Combinatorica*, 26(3):255–275, 2006.
- [12] J. Chuzhoy and S. Khanna. Algorithms for Single-Source Vertex-Connectivity. *Proc. of IEEE FOCS*, October 2008.
- [13] J. Chuzhoy and S. Khanna. An $O(k^3 \log n)$ -Approximation Algorithm for Vertex-Connectivity Survivable Network Design. arXiv.org preprint. arXiv:0812.4442v1 <http://arxiv.org/abs/0812.4442v1>
- [14] J. Fakcharoenphol And B. Laekhanaut. An $O(\log^2 k)$ approximation algorithm for the k -vertex connected subgraph problem. *Proc. of ACM STOC*, 153–158, 2008.
- [15] U. Feige, M. Halldórsson, G. Kortsarz, A. Srinivasan. Approximating the Domatic Number. *SIAM J. Comput.*, 32(1): 172–195, 2002. Preliminary version in *Proc. of ACM STOC*, 134–143, 2000.
- [16] L. Fleischer, K. Jain, D. P. Williamson. Iterative Rounding 2-approximation Algorithms for Minimum-cost Vertex Connectivity Problems. *J. of Computer and System Sciences*, 72(5):838–867, 2006.
- [17] A. Frank. On Connectivity Properties of Eulerian Digraphs. *Annals of Discrete Mathematics*, 41:179–194, 1989.
- [18] A. Frank. Augmenting graphs to meet edge-connectivity requirements. *SIAM J. on Discrete Math.*, 5(1): 22–53, 1992.
- [19] A. Frank, T. Király, and M. Kriesell. On decomposing a hypergraph into k connected sub-hypergraphs. *Discr. Applied Math.*, 131(2): 373–383, 2003.
- [20] M. Grötschel, A. Martin and R. Weismantel. The Steiner tree packing problem in VLSI-design. *Math. Programming* 78: 265–281, 1997.
- [21] H. R. Hind and O. Oellermann. Menger-type results for three or more vertices. *Congressus Numerantium*, 113: 179–204, 1996.
- [22] J. Ivanko. The Weight of a Graph. *Ann. Discr. Math.*, 51: 113–116, 1992.
- [23] K. Jain. A factor 2 approximation algorithm for the generalized Steiner network problem. *Combinatorica*, 21(1):39–60, 2001.
- [24] K. Jain, M. Mahdian, and M. Salavatipour. Packing Steiner Trees. *Proc. of ACM-SIAM SODA*, 266–274, 2003.
- [25] K. Jain, I. I. Mandoiu, V. V. Vazirani, and D. P. Williamson. A Primal-Dual Schema Based Approximation Algorithm for the Element Connectivity Problem. *Proc. of ACM-SIAM SODA*, 484–489, 1999.

- [26] B. Jackson. Some Remarks on Arc-connectivity, Vertex splitting, and Orientation in Digraphs. *Journal of Graph Theory*, 12(3):429–436, 1998.
- [27] T. Király and Lap Chi Lau. Approximate Min-Max Theorems of Steiner Rooted-Orientations of Hypergraphs. *J. of Combin. Theory, Ser. B*, 98(6): 1233–1252, 2008. Preliminary version in *Proc. of IEEE FOCS*, 283–292, 2006.
- [28] G. Kortsarz, R. Krauthgamer, and J.R. Lee. Hardness of Approximation for Vertex-Connectivity Network Design Problems. *SIAM J. on Comput.* 33(3): 704–720, 2004. Preliminary version in *Proc. of APPROX*, 185–199, 2002.
- [29] G. Kortsarz and Z. Nutov. Approximating min-cost connectivity problems. Chapter in *Handbook on Approximation Algorithms and Metaheuristics*, CRC Press, 2006.
- [30] G. Kortsarz and Z. Nutov. Approximating k -node connected subgraphs via critical graphs. *SIAM J. on Computing*, 35(1):247–257, 2005.
- [31] M. Kriesell. Edge-disjoint trees containing some given vertices in a graph. *J. of Combin. Theory, Ser. B*, 88:53–63, 2003.
- [32] Lap Chi Lau. An approximate max-Steiner-tree-packing min-Steiner-cut theorem. *Combinatorica* 27(1), 71–90, 2007. Preliminary version in *Proc. of IEEE FOCS*, 61–70, 2004.
- [33] Lap Chi Lau. Packing Steiner forests. *Proc. of IPCO*, 362–376, 2005.
- [34] László Lovász. On some connectivity properties of Eulerian graphs. *Acta Math. Hung.* 28, 129–138, 1976.
- [35] W. Mader. A reduction method for edge connectivity in graphs. *Ann. Discrete Math.*, 3, 145–164, 1978.
- [36] Zeev Nutov. An almost $O(\log k)$ -approximation for k -connected subgraphs. *Proc. of ACM-SIAM SODA*, 912–921, 2009.
- [37] N. Robertson, P.D. Seymour. Graph Minors. XVII. Taming a Vortex. *J. of Combin. Theory, Ser. B* 77(1): 162–210, 1999.

A Packing Element-Disjoint Trees and Forests

A Counterexample to the Random Coloring algorithm for packing Steiner Forests.

We first define a graph H_k , which we use subsequently. H_k has two black vertices x and y , and k white vertices, each incident to both x and y . (That is, there are k disjoint paths of white vertices from x to y .) Given a graph G , we define the operation of inserting H_k along an edge $pq \in E(G)$ as follows: Add the vertices and edges of H_k to G , delete the edge pq , and add edges from p to x and q to y . (If we collapsed H_k to a single vertex, we would have subdivided the edge pq .) Figure 2 below shows H_4 and the effect of inserting H_4 along an edge.

We now describe the construction of our counterexample. We begin with 2 black vertices s and t , and k vertex-disjoint paths between them, each of length $k + 1$; there are no edges besides the ones just described. Each of the k^2 vertices besides s and t is white. It is obvious that s and t are k -element-connected in this graph. Now, to form our final graph G_k , insert a copy of H_k along each of the $k(k - 1)$ edges between a pair of white vertices. Fig. 3 below shows the construction of G_3 .

The following claims are immediate:

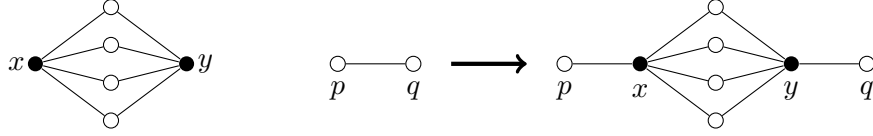


Figure 2: On the left, the graph H_4 . On the right, inserting it along a single edge pq .

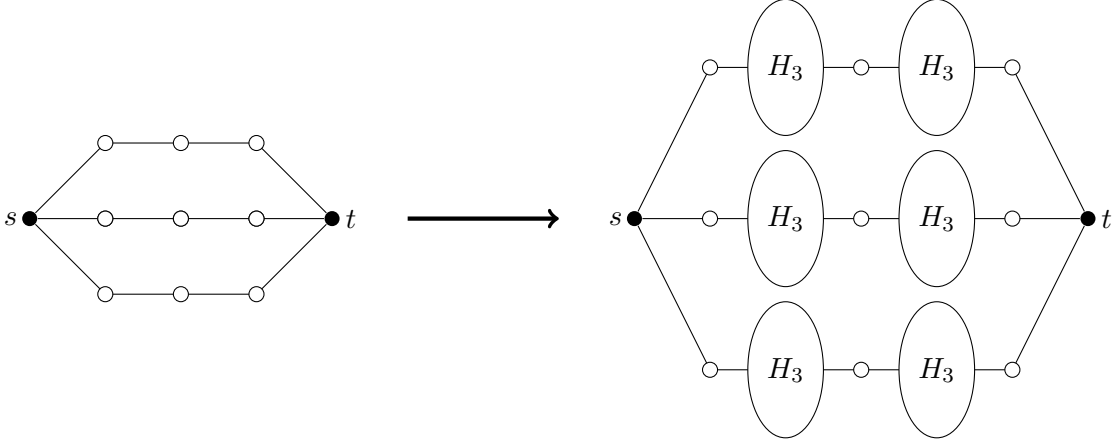


Figure 3: The construction of G_3 .

- The vertices s and t are k -element-connected in G_k .
- For every copy of H_k , the vertices x and y are k -white connected in G_k .
- The graph G_k is bipartite, with the white vertices and the black vertices forming the two parts.

We use G_k as an instance of the Steiner-forest packing problem; s and t form one group of terminals, and for each copy of H_k , the vertices x and y of that copy form a group. From our claims above, each group is k -element-connected.

If we use the algorithm of Cheriyan and Salavatipour, there are no edges between white vertices to be deleted or contracted, so we move directly to the coloring phase. If colors are assigned to the white vertices randomly, it is easy to see that no color class is likely to connect up s and t . The probability that a white vertex is given color i is $\frac{c \log |T|}{k}$, for some constant c . The vertices s and t can be connected iff the same color is assigned to all the white vertices on one of the k paths from s to t in the graph formed from G_k by contracting each H_k to a single vertex. The probability that *every* vertex on such a path will receive the same color is $\left(\frac{c \log |T|}{k}\right)^k$; using the union bound over the k paths gives us the desired result.

A.1 Packing Trees in Planar Graphs

Lemma A.1. *Let $G(T \cup W, E)$ be a planar graph with minimum degree 3, in which W is a stable set. There exists a vertex $t \in T$ of degree at most 10, with at most 5 neighbors in T .*

Proof: Our proof uses the *discharging* technique. Assume, for the sake of contradiction, that every vertex $t \in T$ has degree at least 11, or has at least 6 neighbors in T . By multiplying Euler's formula by 4, we observe

that for a planar graph $G(V, E)$ with face set F , $(2|E| - 4|V|) + (2|E| - 4|F|) = -8$. We rewrite this as $\sum_{v \in V} (d(v) - 4) + \sum_{f \in F} (l(f) - 4) = -8$, where $d(v)$ and $l(f)$ denote the degree of vertex v and length of face f respectively.

Now, in our given graph G , assign $d(v) - 4$ units of *charge* to each vertex $v \in T \cup W$, and assign $l(f) - 4$ units of charge to each face f : Note that the net charge on the graph is negative. (It is equal to -8 .) We describe rules for redistributing the charge through the graph such that after redistribution, if every vertex $t \in T$ has degree at least 11 or has at least 6 neighbors in T , the charge at each vertex and face will be non-negative. But no charge is added or removed (it is merely rearranged), and so we obtain a contradiction.

We use the following rules for distributing charge:

1. Every terminal $t \in T$ distributes $1/3$ unit of charge to each of its neighbors in W .
2. Every terminal $t \in T$ distributes $1/2$ unit of charge to each triangular face f it is incident to, unless the face contains 3 terminals. In this case, it distributes $1/3$ unit of charge to the face.

We now observe that every vertex of W and every face has non-negative charge. Each vertex $u \in W$ has degree at least 3 (the graph has minimum degree 3), so its initial charge was at least -1 . It did not give up any charge, and rule 1 implies that it received $1/3$ from each of its (at least 3) neighbors, all of which are in T . Therefore, u has non-negative charge after redistribution. If a face f has length 4 or more, it already had non-negative charge, and it did not give up any. If f is a triangle, it starts with charge -1 . It is incident to at least 2 terminals, since W is a stable set; we argue that it gains 1 unit of charge, to end with charge 0. From rule 2, if f is incident to 2 terminals, it gains $1/2$ unit from each of them, and if it is adjacent to 3 terminals, it gains $1/3$ unit from each of them.

It remains only to argue that each terminal $t \in T$ has non-negative charge after redistribution. For ease of analysis, we describe a slightly modified version of the discharging in which each terminal loses at least as much charge as under the original rules, and show that each terminal has non-negative charge under the new discharging rules, listed below:

1. Every terminal t gives $1/3$ unit of charge to *every* neighbor.
2. Every terminal $t \in T$ gives $1/3$ unit of charge to each adjacent triangle.
3. Every terminal t gets back $1/3$ unit of charge from each face f such that both t 's neighbors on f are black.

We first prove that every terminal t loses at least as much charge as under the original rules; see also Fig. 4. The terminal t is now giving $1/3$ unit of charge to all its black neighbors, besides giving this charge to its white neighbors. It is giving less charge ($1/3$ instead of $1/2$) to some triangular neighbors, but every triangle is incident to a black vertex t' besides t ; this neighbor of t received an extra $1/3$ unit of charge from t , and it can give $1/6 = 1/2 - 1/3$ to each face incident to the edge $t - t'$. That is, the extra charge of $1/3$ given by t to t' is enough to compensate for the fact that t may give $1/6$ units less charge to the two faces incident to $t - t'$. Finally, note that if both t 's neighbors on some face f are black, the original rules require t to give only $1/3$ unit to f , which it also does under the new rules. However, it has given $1/3$ unit of charge to these two black neighbors, and they do not need to use this to compensate for t giving too little charge to f ; therefore, they may each return $1/6$ unit of charge to t .

We now argue that every terminal has non-negative charge under the new rules. Let $t \in T$ have degree d ; we consider three cases:

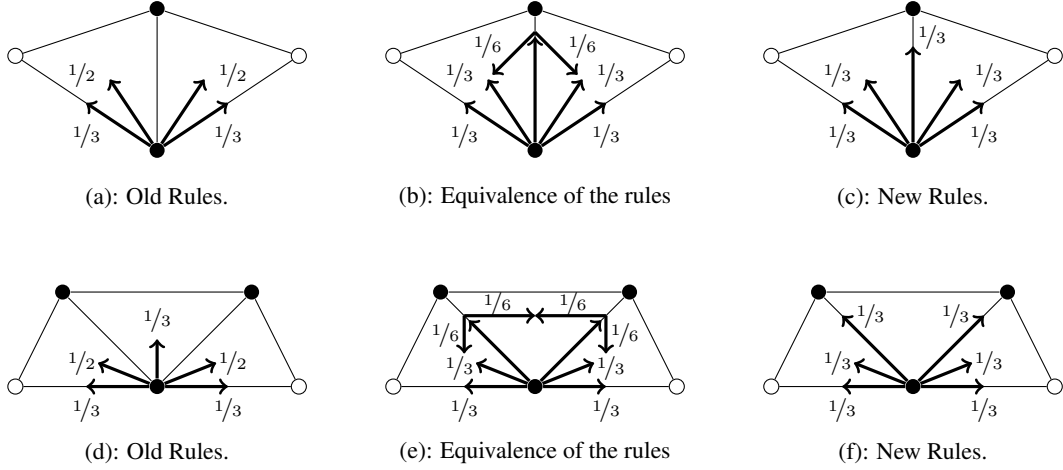


Figure 4: Terminals lose at least as much charge under the new rules.

Part (a) shows the charge given away by a terminal under the original rules, while part (c) shows the charge given away under the new rules; the triangles now receive less charge. Part (b) shows that the extra $1/3$ unit of charge given to the black neighbor under the new rules can be split equally among the two triangles, which has the same effect as giving $1/2$ unit to the triangles. Similarly, part (d) shows the charge given away by a terminal under the original rules, while part (f) shows the charge under the new rule 3: The central triangular face receives $1/3$ unit of charge, but also returns $1/3$ charge to the terminal as both its neighbors on this face are black. Part (e) shows that the extra $1/3$ unit of charge given to each black neighbor under the new rules can be split among the triangles, so the effect is the same as giving $1/3$ unit of charge to the central face, and $1/2$ to each of the other faces.

1. If $d \geq 12$, t gives away $1/3$ to each of its d neighbors and d incident faces, so the total charge it gives away is $2d/3$. (It may also receive some charge, but we ignore this.) Therefore, the net charge on t is $(d - 4) - 2d/3 = (d/3) - 4$; as $d \geq 12$, this cannot be negative.
2. If $d = 11$, we count the number of triangles incident to t . If there are 10 or fewer, t gives away $1/3$ unit of charge to each of its 11 neighbors, and at most $10/3$ to its adjacent triangles, so the net charge on t is at least $(11 - 4) - 11/3 - 10/3 = 0$. If t is incident to 11 triangles, it must be adjacent to at least 6 black vertices, as each triangle incident to t must be adjacent to a black neighbor of t , and no more than 2 triangles incident to t can share a neighbor of t . Since t has degree 11 and at least 6 black neighbors, some pair of black neighbors of t are on a common face, and t must receive $1/3$ unit of charge from this face. It follows that the charge on t is at least $(11 - 4) - 11/3 - 11/3 + 1/3 = 0$.
3. If $d \leq 10$, t has at least 6 black neighbors by hypothesis. It has at most $d - 6$ white neighbors, so there are at least $6 - (d - 6) = 12 - d$ faces f such that both t 's neighbors on f are black. (Delete the white neighbors; there are at least 6 faces incident to t on which both its neighbors are black. When each white vertex is added back, it can only decrease the number of such faces by 1.) The terminal t gives away $1/3$ unit of charge to each of its d neighbors and at most d incident triangles, and receives $1/3$ unit of charge from each face on which both its neighbors are black. Therefore, the net charge on t is at least $(d - 4) - 2d/3 + (12 - d)/3 = 0$. \square

Proof of Lemma 3.6: Our argument is very similar to that of the proof in Section 3.2 that there are two terminals with at least $\lceil k/10 \rceil$ edges between them, except that here we use Lemma A.1 instead of Theorem 3.5.

Let G be the planar multigraph of the reduced instance; every terminal has degree at least k in G . Construct a planar graph G' from G by keeping a single copy of each edge; from Lemma A.1 above, some terminal t has degree at most 10, and at most 5 black neighbors. Let w denote the number of white neighbors of t ,

and b the number of black neighbors. Since each white vertex is incident to only a single copy of each edge in G , there must be at least $\lceil (k-w)/b \rceil$ copies in G of some edge between t and a black neighbor. But $b \leq 5$ and $b+w \leq 10$; it is easy to verify since $k \geq 10$, the smallest possible value of $\lceil (k-w)/b \rceil$ is $\lceil (k-5)/5 \rceil = \lceil k/5 \rceil - 1$. \square

A.2 An Algorithm for Packing Steiner Forests in Planar and Bounded-genus Graphs

For the Planar Steiner Forest Packing problem, we use an algorithm very similar to that for packing Steiner Trees in Section 3.2. Now, as input, we are given sets T_1, \dots, T_m of terminals that are each internally k -connected, but some T_i and T_j may be poorly connected. Precisely as before, as long as each T_i contains at least 2 terminals, Lemma 3.6 is true, so we can contract some pair of terminals t_1, t_2 that have $\lceil k/5 \rceil - 1$ parallel edges between them. Note that if t_1, t_2 are in the same T_i , after contraction, we have an instance in which T_i contains fewer terminals, and we can apply induction. If t_1, t_2 are in different sets T_i, T_j , then after contracting, all terminals in T_i and T_j are pairwise k -connected, so we can merge these two groups into a single set.

In proving the crucial Lemma 3.6, we argued that in the multigraph G of the reduced instance, every terminal has degree at least k (since it is k -element-connected to other terminals), and in the graph G' in which we keep only a single copy of each edge, some terminal has degree at most 10; therefore, there are $\lceil k/10 \rceil$ copies of some edge. However, in the Steiner Forest problem, some T_i may contain only a single terminal t (after several contraction steps). The terminal t may be poorly connected to the remaining terminals; therefore, it may have degree less than k in the multigraph G . If t is the unique low-degree terminal in G' , we may not be able to find a pair of terminals with a large number of edges between them. As a concrete example, consider the graph G_k defined at the beginning of this appendix. (See also Fig. 3, and note that G_k is planar.) We have one terminal set $T_1 = \{s, t\}$, and other sets T_i containing the two terminals of each copy of H_k . After several contraction steps, each copy of H_k may have been contracted together to form a single terminal; each such terminal is only 2-connected to the rest of the graph. In the reduced instance, there is only a single copy of each edge, and Lemma 3.6 does not hold.

We solve this problem by eliminating a set T_i when it has only a single terminal; at this point, we can apply induction and proceed. We formalize this intuition in the following lemma:

Lemma A.2. *Let $G(V, E)$ with a given $T \subseteq V$ be a planar graph, and $t \in T$ be an arbitrary terminal of degree d . Let G' be the graph constructed from G by deleting t , and inserting a $d \times d$ grid of white vertices, with the edges incident to t in G made incident to distinct vertices on one side of the new grid in G' . Then:*

1. G' is planar.
2. For every pair u, v of terminals in G' , $\kappa'_{G'}(u, v) = \kappa'_G(u, v)$.
3. Any set of element-disjoint subgraphs of G' corresponds to a set of element-disjoint subgraphs of G .

Proof Sketch: See Figure 5 showing this operation; it is easy to observe that given a planar embedding of G , one can construct a planar embedding of G' . It is also clear that a set of element-disjoint subgraphs in G' correspond to such a set in G ; every subgraph that uses a vertex of the grid can contain the terminal t .

It remains only to argue that the element-connectivity of every other pair of terminals is preserved. Let u, v be an arbitrary pair of terminals; we show that their element-connectivity in G' is at least their connectivity $\kappa'(u, v)$ in G . Fix a set of $\kappa'(u, v)$ paths in G from u to v ; let \mathcal{P} be the paths that use the terminal t , and let $\ell = |\mathcal{P}|$. We locally modify these ℓ paths in \mathcal{P} by routing them through the grid, so we obtain $\kappa'(u, v)$ element-disjoint paths in G' .

Let \mathcal{P}_u denote the set of prefixes from u to t of the ℓ paths in \mathcal{P} , and let \mathcal{P}_v denote the suffixes from t to v of these paths. Let H denote the $d \times d$ grid that replaces t in G' ; we use \mathcal{P}'_u and \mathcal{P}'_v to denote the corresponding

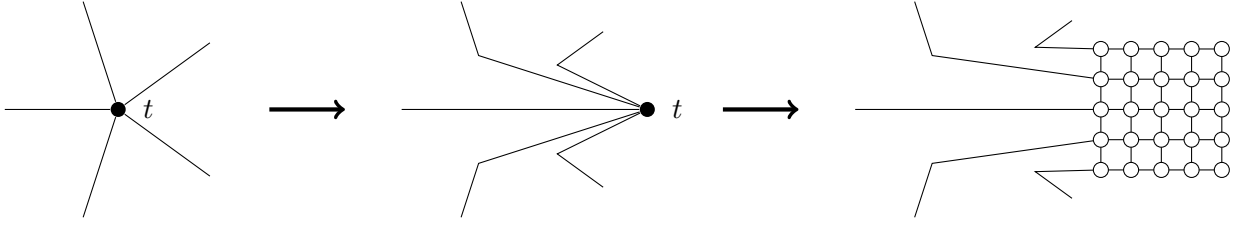


Figure 5: Replacing a terminal by a grid of white vertices preserves planarity and element-connectivity.

paths in G' from u to vertices of H , and from vertices in H to v respectively. Let \mathcal{I} and \mathcal{O} denote the vertices of H incident to paths in \mathcal{P}'_u and \mathcal{P}'_v . It is not difficult to see that there are a set of disjoint paths in the grid H connecting the ℓ distinct vertices in \mathcal{I} to those in \mathcal{O} ; using the paths of \mathcal{P}'_u , together with the paths through H and the paths of \mathcal{P}'_v gives us a set of disjoint paths in G' from u to v . \square

A Counterexample to the existence of 2 terminals with $\Omega(k)$ “Parallel edges” between them: Recall that in the case of planar graphs (or graphs of bounded genus), we argued that there must be two terminals t_1, t_2 with $\Omega(k)$ “parallel edges” between them. (That is, there are $\Omega(k)$ degree-2 white vertices adjacent to t_1 and t_2 .) This is not necessarily the case even in graphs of treewidth 3: The graph $K_{3,k}$, the complete bipartite graph with 3 vertices on one side and k on the other, has treewidth 3. If the three vertices on one side are the terminal set T and the k vertices of the other side are non-terminals, it is easy to see that $\kappa'(T) = k$, but every white vertex has degree 3.

In this example, there are only 3 terminals, so the tree-packing algorithm of Cheriyan and Salavatipour [9] would allow us to find $\Omega(k/\log |T|) = \Omega(k)$ trees connecting them. Adding more terminals incident to all the white vertices would raise the treewidth, so this example does not immediately give us a low-treewidth graph with a large terminal set such that there are few parallel edges between any pair of terminals. However, we can easily extend the example by defining a graph G_m as follows: Let T_1, T_2, \dots, T_m be sets of 2 terminals each, let W_1, W_2, \dots, W_{m-1} each be sets of k white vertices, and let all the vertices in each W_i be adjacent to both terminals in T_i and both terminals in T_{i+1} . (See Fig. 6 below.) The graph G_m has $2m$ terminals, $T = \bigcup_i T_i$ is k -element-connected, and it is easy to verify that G_m has treewidth 4. However, every white vertex has degree 4, so there are no “parallel edges” between terminals. (One can modify this example to construct a counterexample graph G_m with treewidth 3 by removing one terminal from each alternate T_i .)

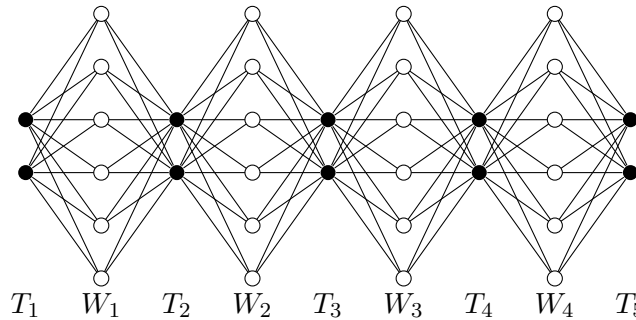


Figure 6: A graph of treewidth 4 with many terminals, but no “parallel edges”.